ON CERTAIN GENERALIZATIONS OF THE PROBLEM OF THREE BODIES.

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The object of the following note is fourfold: first, to determine all the problems of three bodies in which the bodies describe conic sections, under central conservative forces, whatever be the initial conditions of the motion; second, to specialize the preceding solutions, so as to single out those in which the force-function contains only the masses of the bodies and their mutual distances; third, to generalize the latter group to the case in which the orbits are the most arbitrary possible; fourth to generalize the last to the case in which the functions defining the orbits appear in the potential function.

1. If three given particles $(r_1, \theta_1; m_1)$, $(r_2, \theta_2; m_2)$, $(r_3, \theta_3; m_3)$ describe, under central censervative forces, three given coplanar curves whose equations in polar coördinates referred to the center of gravity of the system are

$$(1) \qquad f_1(r_1, \theta_1) = 0, \quad f_2(r_2, \theta_2) = 0, \quad f_3(r_3, \theta_3) = 0,$$

the forces are derived from a potential function which may be written in the form[1]

$$(2) \qquad P = \frac{c^2}{2} \frac{\sum_{i=1}^{3} m_i \left\{ \left(\frac{\partial f_i}{\partial r_i} \right)^2 + \left(\frac{1}{r_i} \frac{\partial f_i}{\partial \theta_i} \right)^2 \right\}}{\left\{ \sum_{i=1}^{3} m_i r_i \frac{\partial f_i}{\partial r_i} \right\}^2},$$

[1] On employing the usual substitutions the form given follows immediately from Oppenheim's solution in rectangular coördinates. See his memoir in the third volume of the Publications of the von Kuffner Observatory.

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where c is the constant in the integral of areas, that is,

$$(3) \qquad c = \sum_{i=1}^{3} m_i r_i^2 \frac{d\theta_i}{dt}.$$

In case the orbits are described independently of the initial conditions, Oppenheim has remarked that it must be possible to throw the function P into the form

$$(4) \qquad P = P_1 + h,$$

where h is a constant independent of the parameters which enter P_1; if such a decomposition of P is impossible, the motion takes place only for special values of the initial constants.

When the orbits are conic sections the equations (1) become

$$(5) \quad f_i(r_i, \theta_i) = r^2_i(A_i \cos^2\theta_i + 2H_i \sin\theta_i \cos\theta_i + B_i \sin^2\theta_i) + 2r_i(G_i \cos\theta_i + F_i \sin\theta_i) - C_i = 0, \quad (i = 1, 2, 3.)$$

If the corresponding functions

$$\frac{\partial f_i}{\partial r_i}, \quad r_i \frac{\partial f_i}{\partial r_i}, \quad \frac{1}{r_i}\frac{\partial f_i}{\partial \theta_i}$$

are constructed, and substituted in the form (2) the latter becomes

$$(6) \quad Q = \frac{c^2}{2} \frac{\begin{array}{r}\sum_{i=1}^{3} m_i\{(H_i^2 - A_i B_i) r_i^2 + 2[(H_i F_i - B_i G_i)\cos\theta_i \\ + (H_i G_i - A_i F_i)\sin\theta_i + F_i^2 + G_i^2 + (A_i + B_i) C_i\}\end{array}}{\left\{\sum_{i=1}^{3} m_i[(G_i\cos\theta_i + F_i\sin\theta_i) r_i - C_i]\right\}^2};$$

this is the most general form of potential function giving rise to conic section trajectories in the problem of three bodies under central conservative forces.

2. From the relations

$$(7) \quad m_i m_j \rho_{ij}^2 = m_i(m_i + m_j) r_i^2 + m_j(m_i + m_j) r_j^2 - m_k^2 r_k^2, \qquad ijk = 123, 231, 312,$$

where ρ_{ij} is the distance between the bodies $(r_i, \theta_i; m_i)$ and $(r_j, \theta_j; m_j)$, it follows that if Q is to be a function of the masses and mutual distances alone we must have

$$(8) \qquad F_i = G_i = 0, \qquad (i = 1, 2, 3).$$

If in addition we have

$$H_1^2 - A_1B_1 = H_2^2 - A_2B_2 = H_3^2 - A_3B_3 = \text{some constant,}$$

$$\text{(9)} \qquad \text{say } \frac{k}{c^2}\left\{\sum_{i=1}^{3} m_i C_i\right\}^2,$$

the function Q may be written

$$\text{(10)} \qquad Q = k\sum_{i=1}^{3} m_i r_i^2 + c^2 \frac{\sum_{i=1}^{3} m_i(A_i + B_i)C_i}{\left\{\sum_{i=1}^{3} m_i C_i\right\}^2} \equiv Q_1 + h.$$

Finally, no noting that the equations (7) lead to the relation

$$\text{(11)} \qquad \left(\sum_{i=1}^{3} m_i\right)\sum_{i=1}^{3} m_i r_i^2 = m_1 m_2 \rho_{12}^2 + m_2 m_3 \rho_{23}^2 + m_3 m_1 \rho_{31}^2,$$

we have Q_1 in the well-known form

$$\text{(12)} \qquad Q_1 = \frac{k}{\sum_{i=1}^{3} m_i}(m_1 m_2 \rho_{12}^2 + m_2 m_3 \rho_{23}^2 + m_3 m_1 \rho_{31}^2),$$

which is thus made to appear as the unique case of conic section orbits for all initial conditions under forces varying as the masses and a function of the mutual distances.

It may be observed here parenthetically that if a similar study be made for the cubic a first condition will be found to demand that the orbits be defined by equations of the form

$$\text{(13)} \quad a_i x_i^3 + 3b_i x_i^2 y_i - 3a_i x_i y_i^2 - b_i y_i^3 - C_i = 0, \quad (i = 1, 2, 3);$$

the remaining analysis of the problem offers no difficulty.

3. Writing

$$\text{(14)} \qquad \frac{\partial f_i}{\partial r_i} = u_i, \quad \frac{\partial f_i}{\partial \theta_i} = v_i,$$

the function (2) becomes

$$\text{(15)} \qquad \frac{c^2}{2}\sum_{i=1}^{3} m_i\left(u_i^2 + \frac{v_i^2}{r_i^2}\right)\Big/\left\{\sum_{i=1}^{3} m_i r_i u_i\right\}^2;$$

considering the case in which

$$(16) \qquad \frac{\sum_{i=1}^{3} m_i\left(u_i^2+\frac{v_i^2}{r_i^2}\right)}{\left\{\sum_{i=1}^{3} m_i r_i u_i\right\}^2} \equiv \frac{\sum_{i=1}^{3} m_i\phi_i(r_i)}{\left\{\sum_{i=1}^{3} m_i\psi_i(r_i)\right\}^2},$$

we find immediately that

$$(17) \qquad u_i=\frac{\phi_i}{r_i}, \quad v_i^2=r_i^2\phi_i-\psi_i^2;$$

and on subjecting these values of u_i and v_i to the condition of integrability we have the following relations

$$(18) \qquad r_i^2\phi_i-\psi_i^2 = \text{some constant, say } \lambda_i^2, \quad (i=1, 2, 3),$$

connecting the functions ϕ_i and ψ_i. The construction of the functions defined by the equations (17) and (18) is effected directly by a simple integration which yields the result that under forces derived from the potential function

$$(19) \qquad R=\frac{c^2}{2}\sum_{i=1}^{3} m_i\left(\frac{\psi_i^2+\lambda_i^2}{r_i^2}\right)\Big/\left\{\sum_{i=1}^{3} m_i\psi_i\right\}^2,$$

three arbitrary masses m_i describe the respective orbits

$$(20) \qquad \int\frac{\psi_i(r_i)}{r_i}dr_i=\pm\lambda_i\theta_i+\mu_i, \qquad (i=1, 2, 3)$$

where the function ψ_i is absolutely arbitrary, and the quantities λ_i, μ_i are any two constants.

In virtue of the relations (7) the function R contains only the masses and mutual distances of the bodies, further, on writing the function ψ_i^2 in the form

$$(21) \qquad \psi_i^2=a_i r_i^2+\omega_i(r_i),$$

where ω_i is an arbitrary function and a_i any constant, it is evident that R can be written in the form (4); whence it follows that the three bodies under forces derived from (19) describe orbits of the form (20) whatever be the initial conditions of the motion of the system.

4. In order to generalize certain of the preceding results further, let us write the equations of the orbits thus

$$(22) \qquad z_i \equiv f_i(x_i, y_i) - c_i = 0, \qquad (i = 1, 2, 3)$$

and the potential function as follows:

$$(23) \qquad P = \frac{c^2}{2} \sum_{i=1}^{3} m_i(p_i^2 + q_i^2) \Big/ \left[\sum_{i=1}^{3} m_i(x_i p_i + y_i q_i) \right]^2,$$

the axes being rectangular about the center of gravity of the system as origin.

Let us consider now the case in which we have

$$(24) \qquad \begin{cases} p_i^2 + q_i^2 = \phi_i(x_i, y_i, z_i; r_i), \\ x_i p_i + y_i q_i = \psi_i(x_i, y_i, z_i; r_i); \end{cases} \quad r_i^2 = x_i^2 + y_i^2 \quad (i = 1, 2, 3),$$

from these it at once follows that

$$(25) \qquad \begin{cases} r_i^2 p_i = x_i \psi_i \pm y_i \sqrt{r_i^2 \phi_i - \psi_i^2}, \\ r_i^2 q_i = y_i \psi_i \mp x_i \sqrt{r_i^2 \phi_i - \psi_i^2}. \end{cases} \qquad (i = 1, 2, 3)$$

The condition of integrability applied to (25) gives

$$(26) \qquad \begin{aligned} r_i^2 \{ 2\phi_i(1 - \psi_{i_{z_i}}) + x_i \phi_{i_{x_i}} + y_i \phi_{i_{y_i}} + r_i \phi_{i_{r_i}} + \psi_i \phi_{i_{z_i}} \} - 2\psi_i(x_i \psi_{i_{x_i}} \\ + y_i \psi_{i_{y_i}} + r_i \psi_{i_{r_i}}) \pm 2(x_i \psi_{i_{y_i}} - y_i \psi_{i_{x_i}}) \sqrt{r_i^2 \phi_i - \psi_i^2} = 0, \\ (i = 1, 2, 3) \end{aligned}$$

an equation whose integration determines ϕ_i when ψ_i is given, and conversely.

(a) In case the functions ϕ_i and ψ_i contain only r_i the equation (26) becomes

$$(27) \qquad \frac{d}{dr_i}(r_i^2 \phi_i - \psi_i^2) = 0,$$

that is to say, it takes the form (18). Accordingly the equations (25) assume the simpler forms

$$(28) \qquad r_i^2 p_i = x_i \psi_i \pm \lambda_i y_i, \quad r_i^2 q_i = y_i \psi_i \mp \lambda_i x_i,$$

whence, by integration, the orbits (20) reappear.

(b) Let ϕ_i be a function only of r_i and ψ_i a function of $n_i z_i$, where n_i is an arbitrary constant; then the condition (26) becomes

$$(29)\qquad r_i\left\{2n_ir_i\phi_i-\frac{d}{dr_i}(r_i^2\phi_i)\right\}=0,\qquad (i=1,2,3),$$

from which we conclude that

$$(30)\qquad \phi_i=\alpha_i^2r_i^{2(n_i-1)},$$

a_i being any constant. The expressions (25) in this case assume the form

$$(31)\qquad \begin{cases} r_i^2p_i=n_ix_iz_i\pm y_i\sqrt{\alpha_i^2r_i^{2(n_i-1)}-n_i^2z_i^2}, \\ r_i^2q_i=n_iy_iz_i\mp x_i\sqrt{\alpha_i^2r_i^{2(n_i-1)}-n_i^2z_i^2}. \end{cases}$$

The determination of the form of z_i from the equations (31) can be affected perhaps most simply in the following manner: That ψ_i is a function only of n_iz_i amounts to saying that

$$(32)\qquad z_i=x_i^{n_i}f_i\left(\frac{y_i}{x_i}\right);$$

substituting the partial derivatives of this function in one or the other of the expressions (31) we obtain the following equation:

$$(33)\qquad \pm\sqrt{\alpha_i^2r_i^{2(n_i-1)}-n_i^2z_i^2}=n_i\frac{y_i}{x_i}f_i-\left(1+\frac{y_i^2}{x_i^2}\right)f_i',$$

whence we have the ordinary differential equation

$$(34)\qquad f_i'=\frac{1}{\xi_i^2+1}\{n_i\xi_if_i\pm\sqrt{\alpha_i^2(\xi_i^2+1)^{n_i}-n_i^2f_i^2}\},\qquad \xi_i=\frac{y_i}{x_i}.$$

The integration of the latter equation may be facilitated by the substitution

$$(35)\qquad 2v_i=n_i\log(\xi_i^2+1),$$

under which (34) takes the form

$$(36)\qquad \frac{df_i}{dv_i}=f_i\pm\frac{1}{n_i\sqrt{e^{\frac{2v_i}{n_i}}-1}}\sqrt{\alpha_i^2e^{2v_i}-n_i^2f_i^2}.$$

Putting now

$$(37)\qquad n_if_i/\alpha_ie^{v_i}=\sin u_i,$$

the equation (36) becomes

$$(38) \qquad \frac{du_i}{dv_i} = \pm \frac{1}{\sqrt{e^{\frac{2v_i}{n_i}} - 1}};$$

whence

$$(39) \qquad u_i = n_i \tan^{-1} \{e^{\frac{2v_i}{n_i}} - 1\}^{\frac{1}{2}} + \beta_i;$$

that is

$$(40) \qquad f_i = \frac{\alpha_i}{n_i} (\xi_i^2 + 1)^{\frac{n_i}{2}} \sin (n_i \tan^{-1} \xi_i + \beta_i);$$

or finally the equations of the orbits become

$$(41) \qquad r_i^{n_i} \sin (n_i \theta_i + \beta_i) = \gamma_i;$$

the corresponding form of the force-function may be written down without difficulty.

If we note that for three equal masses the relations (7) squared give

$$(42) \qquad \rho_{ij}^4 = 4(r_i^4 + r_j^4) + r_k^4, \quad ijk = 123, 231, 312,$$

we see that the solution (41) for $n = 3$ is also a solution of the problem of three equal masses under forces varying as the masses and the cube of the distance.

(c) Let ϕ_i contain both z_i and r_i, while ψ_i is a function only of $n_i z_i$; the condition (26) becomes

$$(43) \qquad 2(1 - n_i)\phi_i + n_i z_i \phi_{i_{z_i}} + r_i \phi_{i_{r_i}} = 0; \quad (i = 1, 2, 3);$$

whence it appears that ϕ_i must have one of the forms

$$(44) \qquad z_i^{\frac{2(n_i-1)}{n_i}} \Phi_i \left(\frac{r_i^{n_i}}{z_i} \right), \quad r_i^{2(n_i-1)} \Psi_i \left(\frac{z_i}{r_i^{n_i}} \right);$$

in case n_i is unity an arbitrary additive constant may be appended to each of these forms. Since Φ_i and Ψ_i are arbitrary functions we have here an infinitude of problems. Considering the second of the forms (32) a little further, the equations (25) become in this case

$$(45) \qquad \begin{aligned} r_i^2 p_i &= n_i x_i z_i \pm y_i \sqrt{r_i^{2n_i} \Psi_i \left(\frac{z_i}{r_i^{n_i}} \right) - n_i^2 z_i^2}, \\ r_i^2 q_i &= n_i y_i z_i \mp x \sqrt{r_i^{2n_i} \Psi_i \left(\frac{z_i}{r_i^{n_i}} \right) - n_i^2 z_i^2}; \end{aligned}$$

and if in particular the symbol Ψ indicates the square we have

$$
(46)\qquad \begin{aligned} r_i^2 p_i &= z_i(n_i x_i \pm y_i\sqrt{1-n_i^2}), \\ r_i^2 q_i &= z_i(n_i y_i \mp x_i\sqrt{1-n_i^2}); \end{aligned}
$$

from which we conclude that the orbits are represented by the equations

$$
(47)\qquad r_i^{n_i} e^{\pm\theta_i\sqrt{1-n_i^2}+\beta_i} = \gamma_i, \qquad (i = 1, 2, 3).
$$

(d) The case in which each of the functions ϕ_i and ψ_i contains both variables r_i and z_i leads to a multitude of problems in which these functions are subject to the single conditions

$$
(48)\qquad r_i[2\phi_i(1-\psi_{i_{z_i}}) + r_i\phi_{i_{r_i}} + \psi_i\phi_{i_{z_i}}] - 2\psi_i\psi_{i_{r_i}} = 0.
$$

(e) If

$$
(49)\qquad \begin{cases} \phi_i = a_i^2 \Big/ \left\{ x_i^{2(\nu_i+n_i)}\left(1+\dfrac{y_i^2}{x_i^2}\right)^{n_i}\omega_i\left(\dfrac{y_i}{x_i}\right)\right\}, \\ \psi_i = (1-\nu_i-n_i)z_i, \end{cases}
$$

where a_i an arbitrary constant, ω_i an arbitrary function, the integration follows a course parallel to that pursued under (b) above, and leads to complicated transcendental equations for the determination of the corresponding orbits.

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